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A DIVERGENT WEIGHTED ORTHONORMAL SERIES OF BROKEN LINE FRANKLIN FUNCTIONS

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ABSTRACT. The purpose of this paper is to define a differentiable function F and an inner product on the space of continuous functions on [0,1] in such a way that the Fourier expansion of F obtained by orthonormalizing the broken line Franklin functions according to this inner product is divergent.

I. **Introduction.** Let $z_1, z_2, \ldots$ denote a reversible number sequence such that $\{z_i\}$ is a subset of [0,1] and is dense in [0,1]. Define $\theta_0, \theta_1, \theta_2, \ldots$ to be the sequence of functions on [0,1] such that $\theta_0(x) = 1$, and for each positive integer i, $\theta_i(x) = 0$ if $0 \leq x \leq z_i$, and $\theta_i(x) = x - z_i$ if $z_i \leq x \leq 1$. Let W denote a continuous strictly increasing function on [0,1]. Define the inner product $((f,g))_W$ of two continuous functions f and g with domain [0,1] to be $\int_0^1 f \cdot g\, dW$. Since no member of the sequence $\theta_0, \theta_1, \theta_2, \ldots$ is a finite linear combination of the other members of the sequence, we can use the Gram-Schmidt process to construct a sequence $\phi_0, \phi_1, \phi_2, \ldots$ of functions on [0,1] such that for each positive integer k, ϕ_k is a linear combination of $\theta_0, \theta_1, \theta_2, \ldots, \theta_k$ and θ_k is a linear combination of $\phi_0, \phi_1, \ldots, \phi_k$ and such that $\phi_0, \phi_1, \phi_2, \ldots$ is orthonormal relative to $((\ ,\ ))_W$. The sequence $\phi_0, \phi_1, \ldots$ will be referred to as the (z, W) sequence. Let I denote the function such that $I(x) = x$.

Franklin [1] showed that if f is a continuous function with domain [0,1], and $z_1, z_2, \ldots$ satisfies a certain property and $\phi_0, \phi_1, \ldots$ is the (z,I) sequence, then the sequence of functions $s_n = \sum_{i=0}^n ((f,\phi_i)) \cdot \phi_i$ converges uniformly to f as n tends to infinity. Wall [3] used (z,W) sequences for arbitrary z and W in the study of certain moment problems and raised the following question to his students. Is it true that if F is a continuous function defined on [0,1] and $z_1, z_2, \ldots$ is a reversible number sequence such that $\{z_i\}$ is a subset of [0,1] and is dense in [0,1] and W is a continuous and strictly increasing function defined on [0,1], then $\sum_{i=0}^n ((f,\phi_i))_W \cdot \phi_i$ converges uniformly to f as n tends to infinity? Sox [2] showed that for every such sequence z, the (z,I) Fourier expansion of a continuous function f converges uniformly to f. It is the purpose of this paper to exhibit a z, W, and F so that if $\phi_0, \phi_1, \phi_2, \ldots$ is the (z,W) sequence, then $\sum_{i=0}^n ((F,\phi_i))\phi_i$ is divergent.

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II. **The example.** There exists a z sequence, a differentiable function F defined on [0,1] and a strictly increasing infinitely differentiable function W defined on [0,1] such that if $\phi_0, \phi_1, \phi_2, \ldots$ is the (z, W) sequence, then the number sequence $\sum_{i=0}^{n} ((\phi_i, F)) \cdot \phi_i(1)$ is unbounded.

Let $(z_1, z_2, \ldots)$ denote the sequence $(0, 1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16, \ldots)$; let K denote the set $\{1/2, 1/4, 1/8, \ldots\}$ and let h denote the function defined by $h(x) = (1-x)^2$. Two sequences $(f_1, f_2, \ldots)$ and $(w_1, w_2, \ldots)$ of functions will now be defined inductively so that the desired functions F and W of the example can be defined by the formulas $F = \sum_{i=1}^{\infty} f_i$ and $W = \sum_{i=1}^{\infty} w_i$. In order to facilitate this construction, fourteen additional sequences, $(N_1, N_2, \ldots)$, $(M_1, M_2, \ldots)$, $(t_1, t_2, \ldots)$, $(g_1, g_2, \ldots)$, $(R_1, R_2, \ldots)$, $(\alpha_1, \alpha_2, \ldots)$, $(\beta_1, \beta_2, \ldots)$, $(\lambda_1, \lambda_2, \ldots)$, $(V_1, V_2, \ldots)$, $(d_1, d_2, \ldots)$, $(s_1, s_2, \ldots)$, $(u_1, u_2, \ldots)$, $(c_1, c_2, \ldots)$ and $(P_1, P_2, \ldots)$, will be defined inductively along with the sequences $(f_1, f_2, \ldots)$ and $(w_1, w_2, \ldots)$.

Set $M_1 = 1/2$ and set $N_1 = 1/2$. Let t_1 denote a member of K such that $t_1 < M_1/4$ and such that the function g_1 defined by the following formulas lies below h.

$$
\begin{aligned}
g_1(x) &= 0, && \text{if } 0 \le x \le N_1,\\
&= 4x/M_1 - 4N_1/M_1, && \text{if } N_1 \le x \le N_1 + t_1,\\
&= -4x/M_1 + 4(N_1 + 2t_1)/M_1, && \text{if } N_1 + t_1 \le x \le N_1 + 2t_1,\\
&= 0, && \text{if } N_1 + 2t_1 < x \le 1.
\end{aligned}
$$

Notice that g_1 is continuous. Let f_1 denote a differentiable function such that if $x \in [0,1]$; then $0 \le f_1(x) \le h(x)$ and if $x \in [0,1] - ([N_1 - t_1/4, N_1 + t_1/4] \cup [N_1 + 3t_1/4, N_1 + 5t_1/4] \cup [N_1 + 7t_1/4, N_1 + 9t_1/4])$, then $f_1(x) = g_1(x)$. Let R_1 denote the class of all straight lines γ with the property that $\gamma(1) < 1$. Set $\alpha_1 = N_1 + t_1/3$, set $\beta_1 = N_1 + 2t_1/3$ and set $\lambda_1 = N_1 + 3t_1$. Let V_1 denote the number set to which the number d belongs if and only if there exists a member γ of R_1 with the property that $d = \int_{\alpha_1}^{\beta_1} [\gamma(x) - f_1(x)]^2\,dx$. Let δ denote a straight line such that $1 < \delta(1) < 2$ and such that δ intersects f_1 at a point (a, b) where $\alpha_1 < a < \beta_1$. If γ is in R_1, then either $\int_{\alpha_1}^{a} [\gamma(x) - \delta(x)]^2\,dx < \int_{\alpha_1}^{a} [\gamma(x) - f_1(x)]^2\,dx$ or else $\int_{a}^{\beta_1} [\gamma(x) - \delta(x)]^2\,dx < \int_{a}^{\beta_1} [\gamma(x) - f_1(x)]^2\,dx$. Therefore, the greatest lower bound of V_1 is positive. Let d_1 denote a positive number less than one and less than the greatest lower bound of V_1. Set $s_1 = 1$. Let u_1 denote a function defined on [0,1] with the following seven properties:

(1) u_1 is infinitely differentiable over [0,1].

(2) $u_1(0) = 0$.

(3) u_1 is strictly increasing over $[0, \lambda_1]$.

(4) If x and y are in $[\lambda_1, 1]$, then $u_1(x) = u_1(y)$.

(5) The restriction of u_1 to $[\alpha_1, \beta_1]$ is a straight line with slope s_1.

(6) If $x \in [0,1]$, then $u_1'(x) \le s_1$.

(7) $[u_1(\alpha_1) - u_1(0)] + [u_1(1) - u_1(\beta_1)] < s_1 \cdot d_1/4$.

There is a positive number $c_1 < 1$ such that for each number x in $[0,1]$, $|c_1 \cdot u_1'(x)| < 1/2$. Let $w_1 = c_1 \cdot u_1$.

Continue this construction inductively for each integer $j > 1$ as follows. Let M_j denote a member of K satisfying the inequalities $\lambda_{j-1} < 1 - 3M_j$ and $M_j < t_{j-1}$. Let $N_j = 1 - M_j$. There is an integer k such that $z_k = N_j$. There exists a $k + 1$ term number sequence $a_0, a_1, \ldots, a_k$ such that if $b_0, b_1, \ldots, b_k$ is a $k + 1$ term number sequence, then

$$\int_0^{\lambda_{j-1}} \left(\sum_{n=1}^{j-1} f_n - \sum_{n=0}^{k} a_n \phi_n \right)^2 d \sum_{n=1}^{j-1} w_n$$

$$\leq \int_0^{\lambda_{j-1}} \left(\sum_{n=1}^{j-1} f_n - \sum_{n=0}^{k} b_n \phi_n \right)^2 d \sum_{n=1}^{j-1} w_n .$$

Let $P_{j-1} = \sum_{n=0}^{k} a_n \phi_n$. Let t_j denote a member of K such that $t_j < M_j/4$ and such that the function g_j, defined by the following formulas, lies below h.

$$\begin{aligned} g_j(x) &= 0, && \text{if } 0 \leq x \leq N_j, \\ &= 4jx/M_j - 4jN_j/M_j, && \text{if } N_j \leq x \leq N_j + t_j, \\ &= -4jx/M_j + 4j(N_j + 2t_j)/M_j, && \text{if } N_j + t_j \leq x \leq N_j + 2t_j, \\ &= 0, && \text{if } N_j + 2t_j < x \leq 1. \end{aligned}$$

Notice that g_j is continuous. Let f_j denote a differentiable function such that if $x \in [0, 1]$, then $0 \leq f_j(x) \leq h(x)$ and if $x \in [0, 1] - \{[N_j - t_j/4, N_j + t_j/4] \cup [N_j + 3t_j/4, N_j + 5t_j/4] \cup [N_j + 7t_j/4, N_j + 9t_j/4]\}$, then $f_j(x) = g_j(x)$. Let R_j denote the class of all straight lines γ with the property that $\gamma(1) < j$. Set $\alpha_j = N_j + t_j/3$, set $\beta_j = N_j + 2t_j/3$, and set $\lambda_j = N_j + 3t_j$. Let V_j denote the number set to which the number d belongs if and only if there exists a member γ of R_j with the property that $d = \int_{\beta_j}^{\alpha_j} (\gamma(x) - f_j(x))^2 dx$. As with V_1, the greatest lower bound of V_j is positive. Let d_j denote a positive number less than d_{j-1} and less than the greatest lower bound of V_j. Set $s_j = c_{j-1} \cdot s_{j-1} \cdot d_{j-1}/(64j^2)$.

Let u_j denote a function defined on $[0,1]$ with the following eight properties:

(1) u_j is infinitely differentiable over $[0,1]$.

(2) If $x \in [0, \lambda_{j-1}]$, then $u_j(x) = 0$.

(3) u_j is strictly increasing on $[\lambda_{j-1}, \lambda_j]$.

(4) If x and y are in $[\lambda_j, 1]$, then $u_j(x) = u_j(y)$.

(5) The restriction of u_j to $[\alpha_j, \beta_j]$ is a straight line with slope s_j.

(6) If $x \in [0, 1]$, then $u_j'(x) \leq s_j$.

(7) $[u_j(\alpha_j) - u_j(0)] + [u_j(1) - u_j(\beta_j)] < s_j \cdot d_j/(64j^2)$.

(8) $u_j(\lambda_{j-1} + M_j) < s_j \cdot d_j/(32j^2(|P_{j-1}(\lambda_{j-1})| + 1)^2)$.

There is a positive number $c_j < 1$ such that for each x in $[0,1]$ and each positive integer $k \leq j$, $|c_j \cdot u_j^{(k)}(x)| < 1/2^j$ where $u_j^{(k)}$ is the kth derivative of u_j. Let $w_j = c_j \cdot u_j$.

Notice that if x is a number in $[0,1)$ and there exists a positive integer i such that $f_i(x) \neq 0$, then there is an open set S containing x such that if j is a positive integer distinct from i and ψ is in S, then $f_j(\psi) = 0$. It follows from this fact that the function $F = \sum_{i=1}^{\infty} f_i$ exists and is differentiable at each number x in $[0,1)$. But for each $x \in [0,1]$, we have that $0 \leq F(x) \leq h(x)$ and therefore F is differentiable at 1.

If m is a positive integer, and j is an integer greater than m, and x is a number in $[1-2M_j, 1]$, then $|W^{(m)}(x)| < 1/2^j$. Notice that this fact implies that W and all of its derivatives exist on $[0,1]$ and $W^{(m)}(1) = 0$.

Suppose that there exists an integer $j > 1$ such that if k is the integer with the property that $z_k = N_j$, then $\sum_{i=0}^{k} ((\phi_i, F))_W \phi_i(1) < j$. Let $G = \sum_{i=0}^{k} ((\phi_i, F))_W \phi_i$. G has the property that if $a_0, a_1, \ldots, a_k$ is a $k+1$ term number sequence, then

$$\int_0^1 (F-G)^2 dW \leq \int_0^1 \left(F - \sum_{i=0}^{k} a_i \phi_i\right)^2 dW.$$

Since $\lambda_{j-1} < N_j < \alpha_j < \beta_j$, we have that

$$\int_0^1 (F-G)^2 dW \geq \int_0^{\lambda_{j-1}} (F-G)^2 dW + \int_{\alpha_j}^{\beta_j} (F-G)^2 dW.$$

But the restriction of F to $[\alpha_j, \beta_j]$ is a subset of f_j and the restriction of G to $[\alpha_j, \beta_j]$ lies on a straight line γ such that $\gamma(1) < j$. Therefore, we have that $\int_{\alpha_j}^{\beta_j} (F-G)^2 dW = \int_{\alpha_j}^{\beta_j} (f_j - \gamma)^2 dW$. But the restriction of W to $[\alpha_j, \beta_j]$ lies on a straight line with slope $c_j \cdot s_j$ and therefore

$$\int_{\beta_j}^{\alpha_j} (f_j - \gamma)^2 dW = c_j \cdot s_j \cdot \int_{\alpha_j}^{\beta_j} (f_j - \gamma)^2 (x)\, dx.$$

Since γ is in R_j, $\int_{\beta_j}^{\alpha_j} (f_j - \gamma)^2 (x)\, dx > d_j$. Combining these inequalities gives the inequality

$$\int_0^1 (F-G)^2 dW > \int_0^{\lambda_{j-1}} (F-G)^2 dW + c_j \cdot s_j \cdot d_j.$$

Let H denote the function with the following four properties:

(1) The restriction of H to $[0, \lambda_{j-1}]$ is P_{j-1}.

(2) The restriction of H to $[\lambda_{j-1}, \lambda_{j-1} + M_j]$ lies on the straight line containing $(\lambda_{j-1}, P_{j-1}(\lambda_{j-1}))$ and $(\lambda_{j-1} + M_j, 0)$.

(3) If $x \in [\lambda_{j-1} + M_j, N_j]$, then $H(x) = 0$.

(4) The restriction of H to $[N_j, 1]$ is a straight line such that if $x \in [\alpha_j, \beta_j]$, then $H(x) = F(x)$.

There is a $k+1$ term number sequence $a_0, a_1, \ldots, a_k$ such that $H(x) = a_0\phi_0 + a_1\phi_1 + \cdots + a_k\phi_k$, and therefore,

$$\text{(A)} \qquad \begin{aligned} \int_0^1 (F-H)^2 dW &\geq \int_0^1 (F-G)^2 dW \\ &> \int_0^{\lambda_{j-1}} (F-G)^2 dW + c_j \cdot s_j \cdot d_j. \end{aligned}$$

Notice that

$$\text{(B)}\quad \begin{aligned}\int_0^1 (F-H)^2\,dW = \int_0^{\lambda_{j-1}} (F-H)^2\,dW + \int_{\lambda_{j-1}}^{\lambda_{j-1}+M_j} (F-H)^2\,dW\\ + \int_{\lambda_{j-1}+M_j}^{\alpha_j} (F-H)^2\,dW + \int_{\alpha_j}^{\beta_j} (F-H)^2\,dW\\ + \int_{\beta_j}^1 (F-H)^2\,dW.\end{aligned}$$

For each $x \in [0,\lambda_{j-1}]$, $F(x) = \sum_{i=1}^{j-1} f_i(x)$ and $H(x) = P_{j-1}(x)$. Therefore, from the definition of P, we obtain:

$$\text{(C)}\quad \begin{aligned}\int_0^{\lambda_{j-1}} (F-H)^2\,dW = \int_0^{\lambda_{j-1}} \left(\left(\sum_{i=1}^{j-1} f_i\right) - P_{j-1}\right)^2 dW\\ \leq \int_0^{\lambda_{j-1}} (F-G)^2\,dW.\end{aligned}$$

For each $x \in [\lambda_{j-1},\lambda_{j-1}+M_j]$, $F(x) = 0$ and $|H(x)| \leq |P_{j-1}(\lambda_{j-1})|$. Moreover,

$$W(\lambda_{ij}+M_j) - W(\lambda_{j-1}) = w_j(\lambda_{j-1}+M_j) - w_j(\lambda_{j-1}) < \frac{c_j \cdot s_j \cdot d_j}{4j(|P_{j-1}(\lambda_{j-1})|+1)^2}.$$

Therefore, we have that

$$\text{(D)}\quad \begin{aligned}\int_{\lambda_{j-1}}^{\lambda_{j-1}+M_j} (F-H)^2\,dW \leq \frac{(P_{j-1}(\lambda_{j-1}))^2 \cdot c_j \cdot s_j \cdot d_j}{(4j(P_{j-1}(\lambda_{j-1})+1)^2)}\\ < \frac{c_j \cdot s_j \cdot d_j}{4}.\end{aligned}$$

For each $x \in [\lambda_{j-1}+M_j,\alpha_j]$, $(F-H)^2(x) < 1$. Moreover, $W(\alpha_j) - W(\lambda_{j-1} + M_j) = w_j(\alpha_j) - w_j(\lambda_{j-1}+M_j) < c_j \cdot s_j \cdot d_j/4$. There inequalities imply

$$\text{(E)}\quad \int_{\lambda_{j-1}+M_j}^{\alpha_j} (F-H)^2\,dW < c_j \cdot s_j \cdot d_j/4.$$

For each $x \in [\alpha_j,\beta_j]$, $F(x) = H(x)$ and therefore,

$$\text{(F)}\quad \int_{\alpha_j}^{\beta_j} (F-H)^2\,dW = 0.$$

For each $x \in [\beta_j,1]$, $(F-H)^2(x) < 16j^2$. Moreover,

$$\begin{aligned}W(1) - W(\beta_j) &= w_j(1) - w_j(\beta_j) + w_{j+1}(1) - w_{j+1}(\beta_j) + w_{j+2}(1) - w_{j+2}(\beta_j) + \cdots\\ &< c_j s_j d_j/(64j^2) + c_{j+1}s_{j+1}\cdot(\lambda_{j+1}-\lambda_j) + c_{j+2}s_{j+2}\cdot(\lambda_{j+2}-\lambda_{j+1}) + \cdots\\ &< c_j s_j d_j/(32j^2).\end{aligned}$$

Therefore, we have the inequality

$$\text{(G)}\quad \int_{\beta_j}^1 (F-H)^2\,dW < (16j^2)(s_j d_j c_j/(32j^2)) < c_j s_j d_j/2.$$

Combining inequalities (B), (C), (D), (E), (F), and (G) yields

$$\int_0^1 (F-H)^2 dW < \int_0^{\lambda_{j-1}} (F-G)^2 dW + c_j s_j d_j$$

which contradicts inequality (A) and therefore the proof is completed.

This example gives rise to a number of questions:

(1) Characterize those strictly increasing continuous functions w which have the property that if f is a continuous function on 0,1 and $z_1, z_2, \ldots$ is a z sequence, then the (z,w) Fourier expansion of f converges uniformly to f.

(2) By a slight modification of the example in this paper, one can construct a z, f, and w such that the (z,w) Fourier expansion of f converges pointwise to f but not uniformly to f. However, the following question remains unanswered: Is there a (z,w) system such that if f is a continuous function on $[0,1]$ then the (z,w) Fourier expansion of f converges pointwise to f, but there exists a continuous g on $[0,1]$ such that the (z,w) Fourier expansion of g does not converge uniformly to g?

(3) Is there a strictly increasing continuous w such that there exist sequences z and z' such that if f is continuous on $[0,1]$ then the (z,w) Fourier expansion of f converges uniformly to f, but there exists a continuous function g on $[0,1]$ such that the (z',w) Fourier expansion of g does not converge to g?

(4) Is the following statement a theorem: If w is a strictly increasing continuously differentiable function on $[0,1]$ and $z_1, z_2, \ldots$ is a z sequence in $[0,1]$ and f is continuously differentiable on $[0,1]$, then the (z,w) Fourier expansion of f converges uniformly to f over $[0,1]$.

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